EDGE MAGICNESS OF *N* COPIES OF GENERALIZED PETERSEN GRAPHS

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Let $G = (V, E)$ be a simple graph with v vertices and e edges. An *edge-magic total labeling* is a bijection $f: V \cup E \to \{1, 2, \ldots v + e\}$ with the property that for each edge xy, $f(x) + f(y) + f(xy) = k$, a constant called the magic sum or magic constant of the graph G. The graph G which admits any edge magic total labeling is called *edge magic total*. A generalized Petersen graph $P(n, m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$ is a 3-regular graph with $2n$ vertices $u_1, u_2, \ldots u_n, v_1, v_2, \ldots v_n$ and edges (u_i, v_i), (u_i, u_{i+1}), (v_i, v_{i+m}) for all $i \in \{1, 2, \ldots n\}$, where the subscripts are taken modulo n. $P(5, 2)$ is the standard Petersen graph. In this paper we study the edge magicness of N copies of generalized Petersen graphs where N is a finite positive integer.

Keywords: Edge-Magic Total Labeling, Copywise Edge Magic, Generalized Petersen Graphs.

INTRODUCTION

A very popular concept of Graph theory is the concept of labeling of graphs which was introduced in the late 1960's. Graph labeling is an assignment of integers to the vertices or edges or both under certain conditions. In [2] J. A. Gallian gives a detailed survey of various graph labelings. In this paper, we study the edge magicness of N copies of generalized Petersen graphs where N is a finite positive integer. Edge magic total labelings were first introduced and investigated by Kotzig & Rosa in 1970 [3].

Definition 1 Let $G = (V, E)$ be a simple graph with v vertices and e edges. An *edge-magic total labeling* is a bijection $f : V \cup E \to \{1, 2, \ldots v + e\}$ with the property that for each edge xy, $f(x) + f(y) + f(xy) = k$, a constant called the magic sum or magic constant of the graph G. The graph G which admits any edge magic total labeling is called *edge magic total*. In particular if $f(V) = \{1, 2, \ldots v\}$, then f is called super edge-magic total labeling and the graph G is called super edge-magic total.

Dual Labeling: If f is an edge-magic total labeling of $G = P(n, m)$ with v vertices and e edges, then its dual labeling f' is defined by $f'(x) = v + e + 1 - f(x) \ \forall \ x \in V$ $f'(xy) = v + e + 1 - f(xy) \ \forall \ xy \in E$ with magic constant $k' = 3(v + e + 1) - k$

Definition 2 A generalized Petersen graph $P(n, m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$ is a 3-regular graph with $2n$ vertices $u_1, u_2, \ldots u_n, v_1, v_2, \ldots v_n$ and edges (u_i, v_i), (u_i, u_{i+1}), (v_i, v_{i+m}) for all $i \in$

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$\{1, 2, \ldots n\}$, where the subscripts are taken modulo n. $P(5, 2)$ is the standard Petersen graph.

Definition 3 Let $G = (V, E)$ be a simple graph with v vertices and e edges. Consider NG, the N copies of G where N is a finite positive integer. A bijection $g : V(NG) \cup E(NG) \to \{1, 2, \ldots N(v + e)\}$ is called *copywise edge magic total labeling* if there exists magic constants $k_1, k_2, \ldots k_N$ such that each of the tth copy of NG becomes edge magic total separately with its own magic constant k_t for $t = 1, 2, \ldots N$ and NG is called copywise edge magic total.

From [1] we get edge magic total labelings of generalized Petersen graphs as given in theorem 1 and theorem 2.

Theorem 1 If n odd, $n \geq 3$, then the generalized Petersen graph $P(n, 1)$ has an edge-magic total labeling with the magic constant (i) $k = \frac{1}{2}(11n + 3)$ (ii) $k = \frac{1}{2}(19n + 3)$.

Proof of (i): Consider $G = (V, E) = P(n, 1)$ with $v = 2n$ vertices and $e = 3n$ edges.

Define $f: V \cup E \to \{1, 2, \ldots v + e = 5n\}$ as follows:

$$f(u_i) = \begin{cases} \frac{1}{2}(4n - i + 1) & for\ i \equiv 1 \pmod 2 \\ \frac{1}{2}(3n - i + 1) & for\ i \equiv 0 \pmod 2 \end{cases};$$

$$f(v_i) = \begin{cases} \frac{1}{2}(n - i) & for\ i \equiv 1 \pmod 2, i \neq n \\ n & for\ i = n \\ \frac{1}{2}(2n - i) & for\ i \equiv 0 \pmod 2 \end{cases},$$

$$f(u_i u_{i+1}) = \begin{cases} 2n+i+1 & for\, i \ne n \\ 2n+1 & for\, i = n \end{cases};$$

$$f(v_i v_{i+1}) = \begin{cases} 4n+2 & for\, i = n \\ 4n+1 & for\, i = n-1 \\ 4n+2+i) & otherwise \end{cases},$$

$$f(u_i v_i) = \begin{cases} 3n+i+1 & for\, i \ne n \\ 3n+1 & for\, i = n. \end{cases}$$

For each edge, the weight is obtained as

$$f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) =$$
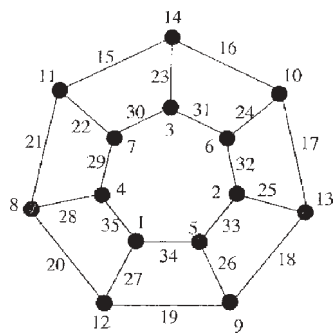
$$f(u_i) + f(u_i v_i) + f(v_i) = \frac{1}{2}(11n + 3)$$

Thus f is an edge-magic total labeling. Moreover $f(V)$ = {1, 2, ... 2n}. Hence this is also a super edge-magic total labeling.

Proof of (ii):

By duality, $P(n, 1)$ has an edge-magic total labeling with magic constant

$$k' = 3(2n + 3n + 1) - \frac{1}{2}(11n + 3) = \frac{1}{2}(19n + 3)$$

Example

Edge-Magic Total Labeling of $P(7, 1)$ with $k = \frac{1}{2}(11n + 3) = 40$

Theorem 2 *If n odd, $n \ge 5$, then the generalized Petersen graph $P(n, 2)$ has an edge-magic total labeling with the magic constant $k = \frac{1}{2}(15n + 3)$.*

Proof: Consider $G = (V, E) = P(n, 2)$ with $v = 2n$ vertices and $e = 3n$ edges.

Define $h: V \cup E \to \{1, 2, ... v + e = 5n\}$ as follows:

$$h(u_i) = \begin{cases} \frac{1}{2}(2n+i+1) & for\, i \equiv 1 (mod\,2) \\ \frac{1}{2}(3n+i+1) & for\, i \equiv 0 (mod\,2) \end{cases};$$

$$f(u_i u_{i+1}) = \begin{cases} 2n+i+1 & for\, i \ne n \\ 2n+1 & for\, i = n \end{cases};$$

$$h(v_i v_{i+2}) = \begin{cases} \frac{1}{2}(i+3) & for\, i \equiv 1 (mod\,2) \\ \frac{1}{2}(n+i+3) & for\, i \equiv 2 (mod\,2), i \ne n-1 \\ 1 & for\, i \equiv n-1. \end{cases}$$

Case 1 : For $n \equiv 1 \pmod 4$

$$h(v_i) = \begin{cases} \frac{1}{4}(16n-i+1) & for\ i \equiv 1 (mod\,4), \\ \frac{1}{4}(13n-i+1) & for\ i \equiv 2 (mod\,4), \\ \frac{1}{4}(14n-i+1) & for\ i \equiv 3 (mod\,4), \\ \frac{1}{4}(15n-i+1) & for\ i \equiv 0 (mod\,4). \end{cases};$$

$$h(u_i v_i) = \begin{cases} \frac{1}{4}(10n-i+3) & for\ i \equiv 1 (mod\,4), \\ \frac{1}{4}(11n-i+3) & for\ i \equiv 2 (mod\,4), \\ \frac{1}{4}(12n-i+3) & for\ i \equiv 3 (mod\,4), \\ \frac{1}{4}(9n-i+3) & for\ i \equiv 0 (mod\,4). \end{cases}$$

Case 2 : For $n \equiv 3 \pmod 4$

$$h(v_i) = \begin{cases} \frac{1}{4}(16n-i+1) & for\ i \equiv 1 (mod\,4), \\ \frac{1}{4}(15n-i+1) & for\ i \equiv 2 (mod\,4), \\ \frac{1}{4}(14n-i+1) & for\ i \equiv 3 (mod\,4), \\ \frac{1}{4}(13n-i+1) & for\ i \equiv 0 (mod\,4). \end{cases};$$

$$h(u_i v_i) = \begin{cases} \frac{1}{4}(10n-i+3) & for\ i \equiv 1 (mod\,4), \\ \frac{1}{4}(9n-i+3) & for\ i \equiv 2 (mod\,4), \\ \frac{1}{4}(12n-i+3) & for\ i \equiv 3 (mod\,4), \\ \frac{1}{4}(11n-i+3) & for\ i \equiv 0 (mod\,4). \end{cases}$$
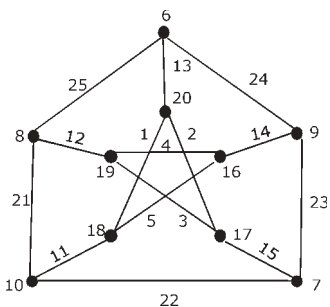
For each edge, the weight is obtained as

$$h(u_i) + h(u_i u_{i+1}) + h(u_{i+1}) = h(v_i) + h(v_i v_{i+1}) + h(v_{i+1}) =$$

$$h(u_i) + h(u_i v_i) + h(v_i) = \frac{1}{2}(15n + 3)$$

Thus h is an edge-magic total labeling. Here h is self-dual.

Example

Edge Magic Total Labeling of $P(5, 2)$ with $k = \frac{1}{2}(15n + 3) = 39$

Copywise Edge Magic Labelings of N$P(n, m)$

Theorem 3 If n odd, $n \geq 3$ and N is a finite positive integer then $NP(n, 1)$ is copywise edge-magic total with the magic constant of the tth copy given by

$$k_t = \frac{[10(N+t)-9]n+3}{2}.$$

Proof: Label the first graph of $NP(n, 1)$ as given in theorem 1. Label each of the tth graph (copy) by adding each label with $(t-1)5n$. Now interchange the corresponding edge labels of the outer cycle as well as the inner cycle and that of the $u_i v_i$ edges in the first graph with the ones in the Nth graph. Do similarly between the second graph and the $(N-1)^{\text{th}}$ graph, the third graph and the $(N-2)^{\text{nd}}$ graph and so on, except in the case of N being odd where the labels of $(N + 1) / 2^{\text{nd}}$ graph remains unaltered.

In this manner, we construct the labeling g of $NP(n, 1)$ as follows: Consider the labeling f given in theorem 1. Then the labeling of the tth graph (copy) is given by

$$g^t(u_i) = f(u_i) + (t-1)5n$$

$$g^t(v_i) = f(v_i) + (t-1)5n$$

$$g^t(u_i v_i) = f(u_i v_i) + (N-t)5n$$

$$g^t(u_i u_{i+1}) = f(u_i u_{i+1}) + (N-t)5n$$

$$g^t(v_i v_{i+1}) = f(v_i v_{i+1}) + (N-t)5n, \text{ where } t = 1, 2, \ldots N.$$

It can be easily verified that the magic constant of the tth graph (copy) is given by $k_t = \dfrac{[10(N+t)-9]n+3}{2}$. Thus $NP(n, 1)$ is copywise edge-magic total.

By duality principle we have the following corollary.

Corollary If n odd, $n \geq 3$ and N is a finite positive integer then $N P(n, 1)$ is copywise edge-magic total with the magic constant of the tth copy given by

$$k'_t = \frac{[10(2N-t)+9]n+3}{2}.$$

Theorem 4 If n odd, $n \geq 3$ and N is a finite positive integer then $N P(n, 2)$ is copywise edge-magic total with the magic constant of the tth copy given by

$$k_t = \frac{[10(N+t)-5]n+3}{2}.$$

Proof: As explained in theorem 3, similarly, we construct the labeling g of $NP(n, 2)$ as follows:

Consider the labeling h given in theorem 2 .Then the labeling of the tth graph (copy) is given by

$$g^t(u_i) = h(u_i) + (t-1)5n$$

$$g^t(v_i) = h(v_i) + (t-1)5n$$

$$g^t(u_i v_i) = h(u_i v_i) + (N-t)5n$$

$$g^t(u_i u_{i+1}) = h(u_i u_{i+1}) + (N-t)5n$$

$$g^t(v_i v_{i+1}) = h(v_i v_{i+1}) + (N-t)5n, \text{ where } t = 1, 2, \ldots N.$$

It can be easily verified that the magic constant of the tth graph (copy) is given by $k_t = \dfrac{[10(N+t)-5]n+3}{2}$. Thus $N P(n, 2)$ is copywise edge-magic total.
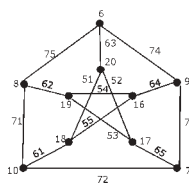
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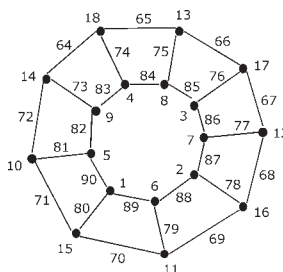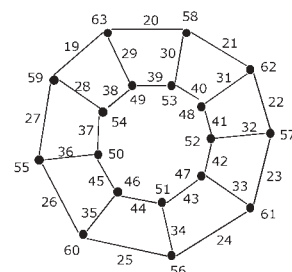
$$k_1 = \frac{1}{2}(21n+3) = 96 \qquad k_2 = \frac{1}{2}(31n+3) = 141$$
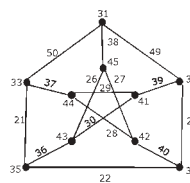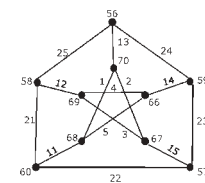
Copywise Edge Magic Total Labeling of 2 $P(9, 1)$

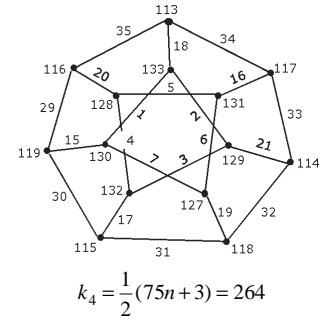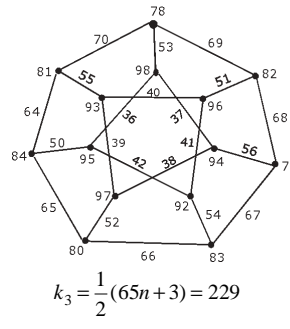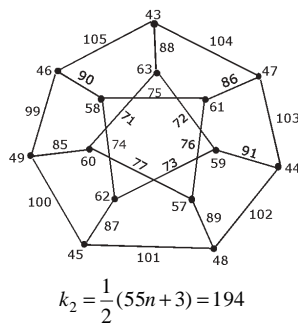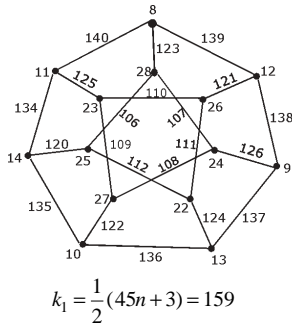
$$k_1 = \frac{1}{2}(35n+3) = 89 \qquad k_2 = \frac{1}{2}(45n+3) = 114 \qquad k_3 = \frac{1}{2}(55n+3) = 139$$
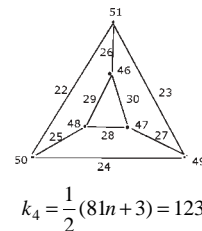
Copywise Edge Magic Total Labeling of 3 $P(5, 2)$

$$k_1 = \frac{1}{2}(45n+3) = 159 \qquad k_2 = \frac{1}{2}(55n+3) = 194 \qquad k_3 = \frac{1}{2}(65n+3) = 229 \qquad k_4 = \frac{1}{2}(75n+3) = 264$$

Copywise Edge Magic Total Labeling of 4 $P(7, 2)$

$$k_1 = \frac{1}{2}(51n+3) = 78 \qquad k_2 = \frac{1}{2}(61n+3) = 93 \qquad k_3 = \frac{1}{2}(71n+3) = 108 \qquad k_4 = \frac{1}{2}(81n+3) = 123 \qquad k_5 = \frac{1}{2}(91n+3) = 138$$
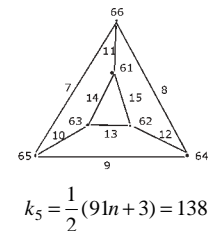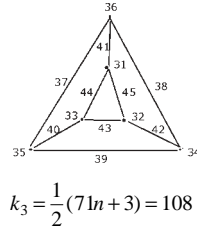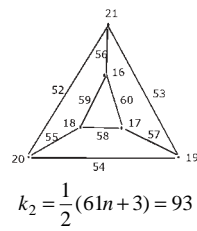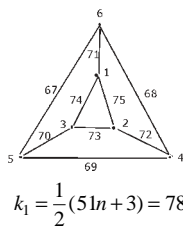
Copywise Edge Magic Total Labeling of 5 $P(3, 1)$

Conclusion

In this paper we have discussed about edge magicness of N copies of generalized Petersen graphs which are isomorphic to each other. In the case of non isomorphic disconnected graphs still the problem of constructing edge magic labelings remains open.

Future Work

As we have constructed edge magic labelings for N copies of generalized Petersen graphs, we look forward to construct consecutive magic labelings for the same.

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